Problem Section 4

Edited by Manjil P. Saikia

School of Mathematics, Cardiff University, CF24 4AG, UK

E-mail: `manjil@saikia.in`

This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *20 December, 2021.* If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 10. *Proposed by B. Sury (Indian Statistical Institute, Bengaluru).*

Prove that

$$\sum_{i=1}^{n} \binom{n}{i} \frac{(-1)^{i-1}}{i} = \sum_{k=1}^{n} \frac{1}{k}.$$

More generally, show that

$$\sum_{1 \le i_1 \le \le \cdots \le i_r \le n} \binom{n}{i_r} \frac{(-1)^{i_r - 1}}{i_1 i_2 \cdots i_r} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

Problem 11. *Proposed by Anupam Saikia (Indian Institute of Technology Guwahati).*

Let S_{1111} be the permutation group on the set $T = \{1, 2, \ldots, 1111\}$. Let G be an abelian subgroup of S_{1111} of order 1313. Show that there must be an integer $i \in T$ such that $\sigma(i) = i$ for all $\sigma \in G$. (*The question requires knowledge of permutation group S_3 and Sylow theorems.*)

Solutions to Old Problems

We received correct solutions to Problems 4, 5 and 6 from **Amit Kumar Basistha** (Anundoram Borooah Academy, Pathsala, India). Problems 7, 8 and 9 of the previous issue are still open to the floor for solutions.

Solution 1. *Solved by the proposer. The solution below is by Anupam Saikia (Indian Institute of Technology Guwahati).*

Observe that $m(a^2-1) = a^{2p}-1 \equiv a^2-1 \pmod p$ by Fermat's Little Theorem. Since $p \nmid (a^2-1)$, we have $p \mid m$. Clearly, m is even. Therefore, $2p \mid m$. Now,

$$a^{2p}-1 = m(a^2-1) \equiv 0 \pmod m \implies a^m \equiv 1 \pmod m.$$

Solution 2. *Solved by the proposer. The remarks below are by Manjil P. Saikia (Cardiff University).*

The problem is a special case of a result of Mahanta and Saikia (see: Theorem 2.2 of 'Refinement of some partition identities of Merca and Yee' (2021), `https://manjilsaikia.in/publ/papers/PJM_MY.pdf`.)

Solution 3. *Solved by the proposer. The solution below is by B. Sury (Indian Statistical Institute, Bangaluru).*

From the first conditions, $f(1,x) = 1 = f(-1,x) = f(x,1) = f(x,-1)$ for all $x \neq 0$. Now, $f(1/x,y)f(x,y) = f(1,y) = 1 = f(x,1) = f(x,1/y)f(x,y)$. Cancelling off $f(x,y)$, we get

$$f(1/x,y) = f(x,1/y) = 1/f(x,y).$$

So, for $x \neq 0,1$, we have by the second condition that

$$1 = f(1/x, 1-1/x) = f\left(x, \frac{1}{1-\frac{1}{x}}\right).$$

Multiplying with $1 = f(x,1-x)$, we get

$$1 = f(x,1-x)f\left(x, \frac{1}{1-\frac{1}{x}}) = f(x, \frac{1-x}{1-\frac{1}{x}}\right) = f(x,-x).$$

Also, then for $x \neq 0,1$, $f(x,x) = f(x,-x)f(x,-1) = 1$. As $f(x,x) = f(x,-x) = 1$ holds for $x=1$ also, this proves (a).

The part (b) follows by noting

$$1 = f(xy,xy) = f(x,xy)f(y,xy) = f(x,x)f(x,y)f(y,x)f(y,y) = f(x,y)f(y,x).$$

Solution 4. *Solved by Amit Kumar Basistha (Anundoram Borooah Academy, Pathsala) and the proposer. The solution below is by Anupam Saikia (Indian Institute of Technology Guwahati).*

The solution exploits a relation between the Fibonacci sequence and the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, *together with some elementary group theory).*

Observe that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for $n \geq 2$ (which can be easily proved by induction). Moreover, A can be considered as an element in the group G consisting of 2×2 matrices of determinant ± 1 with entries from the field $\mathbb{Z}_p$ of p elements. The order of the group G is $2p(p^2-1)$ (G contains $SL_2(\mathbb{Z}_p)$ as a subgroup of index 2). Then, $A^{2p(p^2-1)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod p$, as $ord(A) \mid ord(G)$ and it yields the desired result.

The solution by Amit Kumar Basistha used Binet's form of Fibonacci numbers and elementary number theoretic techniques.

Solution 5. *Solved by Amit Kumar Basistha (Anundoram Borooah Academy, Pathsala). The solution below is by Amit Kumar Basistha (Anundoram Borooah Academy, Pathsala).*

The answer is 'yes'. Consider $n = 4a^2 + 2a + 1$ for some positive integer a. Then $n^2 + 1 = 2(4a^2+1)(2a^2+2a+1)$. If $4a^2+1 = 2a^2+2a+1$ then a is either 0 or 1. So, the two factors $4a^2+1$ and $2a^2+2a+1$ of n are distinct, odd and less than n for $a > 1$. We, thus, have infinitely many values of n for which n^2+1 divides $n!$.

Solution 6. *Solved by the proposer and partially by Amit Kumar Basistha (Anundoram Borooah Academy, Pathsala). The solution below is by B. Sury (Indian Statistical Institute, Bengaluru).*

Note that $q'(\alpha_i) = \prod_{j \neq i}(\alpha_i - \alpha_j)$. We claim that actually the polynomial

$$p(x) = \sum_i p(\alpha_i) \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j}.$$

To see this, consider

$$h(x) = \sum_i p(\alpha_i) \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j} - p(x).$$

Now $h(\alpha_i) = 0$ for all $i = 1, \ldots, n$. But $h(x)$, clearly, has degree at most $(n-1)$. Thus, $h(x)$ must be the zero polynomial. In other words, the claim that $p(x) = \sum_i p(\alpha_i) \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j}$ is proved. As $p(x)$ has degree at the most $n-2$, the right hand side of the above expression must have its coefficient of x^{n-1} to be zero.

The statement also follows simply from the sum of residues theorem applied to the meromorphic function $p(z)/q(z)$.